Conformal Invariance and Near-extreme Rotating AdS Black Holes

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We obtain retarded Green's functions for massless scalar fields in the background of near-extreme, near-horizon rotating charged black holes of five-dimensional minimal gauged supergravity. The radial part of the (separable) massless Klein-Gordon equation in such general black hole backgrounds is Heun's equation, due to the singularity structure associated with the three black hole horizons. On the other hand, we find the scaling limit for the near-extreme, near-horizon background where the radial equation reduces to a hypergeometric equation whose $SL(2,\mathbf{R})^2$ symmetry signifies the underlying two-dimensional conformal invariance, with the two sectors governed by the respective Frolov-Thorne temperatures.

PACS numbers: 04.70.Dy, 11.25.-w, 04.65.+e, 04.50.-h

I. INTRODUCTION

Recently, important new insights into microscopics of extreme rotating black holes (with zero Hawking temperature) have been obtained [1] via a correspondence between these classical gravitational objects and the underlying boundary conformal invariance (Kerr/CFT correspondence). These new insights are important for several reasons; one is that such rotating black holes could be astrophysical: i.e., Kerr black holes close to extremality could exist in our universe, and their microscopics can be addressed via Kerr/CFT correspondence [1].

Employing the boundary conformal invariance techniques [2, 3] allows for the microscopic calculation of extreme rotating black hole entropy which is in agreement with the Bekenstein-Hawking entropy, as first calculated for extreme Kerr black holes in four-dimensions [1]. Numerous follow-up papers confirmed the agreement for other extreme rotating black holes. Specifically, this has been verified for extreme four-dimensional Kerr-Newman (AdS) black holes [4] and for large classes [5] of extreme multicharged rotating black holes in asymptotically Minkowski and AdS space-times, and in diverse dimensions.

Another important insight into internal structure of black holes is obtained by studying probe particles in these black hole backgrounds. It was shown in the past that the wave equation for the massless scalar field is separable and it has amazing simplifications [6, 7], even for general, multicharged rotating black holes in asymptotic Minkowski space-times [8, 9]. In particular, when certain terms can be neglected the radial part of the wave equations has an $SL(2,\mathbf{R})^2$ symmetry, and thus underlying two-dimensional conformal symmetry. This is the case for special black hole backgrounds, such as the near-extreme limit, with a microscopic description in terms of the AdS/CFT correspondence [6, 10], and the near-extreme rotating limit, with a recent microscopic calculation in terms of the Kerr/CFT correspondence [1, 11]. In addition, this conformal symmetry emerges for low-energy scattering [6, 14], as well as in the super-radiant limit [11–13].

On the other hand, for general black hole backgrounds and for scattering at arbitrary energies the wave equation has no $SL(2,\mathbf{R})^2$ symmetry, thus this is a stumbling block for a conformal field theory interpretation in the general case. However, a recent proposal referred to as "hidden conformal symmetry" proposes that the conformal symmetry, as suggested by the massless scalar field wave equation, is still useful [14], only that it is spontaneously broken. This approach has been further studied by a number of researchers, including [15, 16]. For a review, see [17].

In a recent development, in [18] an explicit "subtracted geometry" with manifest $SL(2,\mathbf{R})^2$ conformal invariance was obtained for general rotating black holes in five-dimensional asymptotically flat space-time, by removing certain ambient flat space-time terms in an overall warp factor of the black hole solution. The construction preserves all near-horizon properties of the black holes, such as the thermodynamic potentials and the entropy. The warp factor subtraction provides an explicit realization of the conformal symmetry at the level of the black hole metric. Furthermore, a quantitative relation to the standard AdS/CFT correspondence is obtained by embedding the subtracted black hole geometry in auxiliary six dimensions, resulting in a long rotating string with a fibered $AdS_3 \times S^3$ geometry[19, 20].

In this paper we advance the study of internal structure and emergence of conformal invariance for charged rotating black holes in *asymptotically anti-de Sitter space-times.* While the wave equations for black holes in these

backgrounds are again separable, the radial equation is in general Heun's equation, due to more than two horizons of such black holes, and thus the structure for such general background, does not exhibit $SL(2,\mathbf{R})^2$ conformal invariance. (Note also, that the study of microscopics for black holes in asymptotically anti-de Sitter space-times remains a complex problem.) The goal of our analysis is more modest: We would like to demonstrate conformal invariance for the massless scalar field wave equation in the background of both extreme *and* near-extreme backgrounds of asymptotically AdS black holes.

For concreteness we focus on general charged rotating black holes in five-dimensional minimal supergravity [21]. These black holes are specified by mass, charge, two angular momenta, and cosmological constant.[1]

Specifically, we demonstrate that both the extreme and near-extreme limit of these black holes results in massless scalar wave equations whose solutions are hypergeometric functions. (As a prerequisite technical result for these studies we formulate a precise scaling limit of these near-extreme, near-horizon backgrounds.) We calculate explicitly the retarded Green's functions both for the extreme and near-extreme backgrounds, thus generalizing the Green's function results obtained for four-dimensional Kerr backgrounds [12, 16]. The left- and right-moving sectors of boundary conformal theories are governed by a combination of two Frolov-Thorne temperatures [29], specified along the two combinations of the azimuthal angles. These results lend further support for the underlying conformal field theory description of both extreme and near-extreme rotating black holes in asymptotically anti-de Sitter space-times. (We expect that results, analogous to those in this paper, would be obtained for other multicharged AdS rotating black holes in four, five, and possibly other dimensions, e.g., black holes of [22] and references therein.)

The paper is organized in the following way: In the second section, we present the radial part of the massless scalar field equation (massless Klein-Gordon equation) with a "natural" coordinate transformation, and demonstrate explicitly that the pole structure is governed by the location of the three horizons and each residuum by the surface gravity at each horizon. The resulting equation is Heun's equation.[2] In the third section, we turn to the extreme black hole background. The metric in the extreme case, near-horizon regime was studied in [5] and [28] by taking a specific scaling limit of the general solution. Here we analyze the Klein-Gordon equation, which is a hypergeometric equation with underlying conformal symmetry, and obtain the explicit retarded Green's functions there. In the fourth section, we address the near-extreme and near-horizon limit by taking a scaling limit where an effective dimensionless near-extremality parameter is kept explicitly. The radial equation remains a hypergeometric equation and explicit retarded Green's functions there. In the fifth section, we summarize the results and comment on future directions. In the Appendix, a further analysis of the radial equation with a dimensionless coordinate is presented. We also perform an expansion both for the case of a small cosmological constant and for a small (non-extremality) parameter.

II. THE GENERAL METRIC

The metric for general black holes in five-dimensional minimal gauged supergravity was given in Chong et al. [21] and it can be written in the following form:

$$
\begin{aligned}
ds^2 = &-\left(dt-\frac{a\sin^2\theta}{\Xi_a}\,d\phi-\frac{b\cos^2\theta}{\Xi_b}\,d\psi\right)\\
&\times\left[f\left(dt-\frac{a\sin^2\theta}{\Xi_a}\,d\phi-\frac{b\cos^2\theta}{\Xi_b}\,d\psi\right)\right.\\
&\left.+\frac{2Q}{\Sigma}\left(\frac{b\sin^2\theta}{\Xi_a}\,d\phi+\frac{a\cos^2\theta}{\Xi_b}\,d\psi\right)\right]\\
&+\Sigma\left(\frac{r^2dr^2}{\Delta_r}+\frac{d\theta^2}{\Delta_\theta}\right)+\frac{\Delta_\theta\sin^2\theta}{\Sigma}\left(a\,dt-\frac{r^2+a^2}{\Xi_a}\,d\phi\right)^2\\
&+\frac{\Delta_\theta\cos^2\theta}{\Sigma}\left(b\,dt-\frac{r^2+b^2}{\Xi_b}\,d\psi\right)^2+\frac{1+r^2G}{r^2\Sigma}\\
&\times\left(ab\,dt-\frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}\,d\phi-\frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}\,d\psi\right)^2 \qquad (1)
\end{aligned}
$$

where,

$$
\begin{aligned}
f &= \frac{\Delta_r-2ab\,q-q^2}{r^2\Sigma}+\frac{q^2}{\Sigma^2},\\
\Sigma &= r^2+a^2\cos^2\theta+b^2\sin^2\theta,\\
\Xi_a &= 1-a^2G\ ,\quad \Xi_b=1-b^2G,\\
\Delta_r &= \left(r^2+a^2\right)\left(r^2+b^2\right)\left(1+r^2G\right)\\
&\quad +2ab\,q+q^2-2mr^2,\\
\Delta_\theta &= 1-G(a^2\cos^2\theta-b^2\sin^2\theta). \qquad (2)
\end{aligned}
$$

[1] The uniqueness of the full solution and the separability of the wave equation for the massless scalar field was analyzed by various groups [23–25]. The quasinormal modes and superradiant instability analysis also exists in the literature [26] for this metric.

[2] The wave equation thus indicates that all three horizons play a role in the internal structure of the black hole. A complementary recent result lends support to this picture, by demonstrating that the product of areas of all three horizons is quantized and moduli independent, i.e. it depends explicitly only on the quantized charge, the two angular momenta and the cosmological constant [27].

Here, we should note that we define the parameter $G \equiv g^2$, related to the cosmological constant $\Lambda = -6G$. The "bare" black hole parameters m, q, a, b, and G specify the physical parameters mass M, charge Q and two angular momenta in the following way:

$$M = \frac{m\pi(2\Xi_a + 2\Xi_b - \Xi_a\,\Xi_b) + 2\pi qabG(\Xi_a + \Xi_b)}{4\Xi_a^2\,\Xi_b^2}\,, \tag{3}$$

$$Q = \frac{\sqrt{3}\,\pi q}{4\Xi_a\,\Xi_b}\,, \tag{4}$$

$$J_a = \frac{\pi[2am + qb(1 + a^2G)]}{4\Xi_a^2\,\Xi_b}\,, \tag{5}$$

$$J_b = \frac{\pi[2bm + qa(1 + b^2G)]}{4\Xi_b^2\,\Xi_a}\,. \tag{6}$$

The Klein-Gordon equation for a massless scalar field is given by,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) = 0, \tag{7}$$

where g is the determinant of the metric. The scalar field Ansatz can be written as,

$$\Phi = e^{-i\omega t + im_1\phi + im_2\psi}\tilde{R}(r)F(\theta). \tag{8}$$

For the sake of simplicity, the angular eigenvalues will be set to zero ($m_1 = m_2 = 0$) in this section. (For non-zero $m_{1,2}$ parameters the wave equation exhibits analogous pole structure, but with the residua are corrected accordingly. For general asymptotically flat black holes this has been done explicitly in [6].)

The radial equation then reads,

$$\frac{\Delta_r}{r}\frac{d}{dr}\left(\frac{\Delta_r}{r}\frac{d\tilde{R}}{dr}\right) + \Bigg\{\frac{[(r^2 + a^2)(r^2 + b^2) + abq]^2\omega^2}{r^2} - \Delta_r\left(c_0 + \frac{a^2b^2}{r^2}\omega^2\right)\Bigg\}\tilde{R} = 0, \tag{9}$$

where c_0 is the separation constant, which is the eigenvalue of the angular equation, displayed in the next subsection.

As the horizon equation Δ_r is quadratic in r, a coordinate transformation $u = r^2$ renders the equation in the following form:

$$\frac{d}{du}\left(\Delta_u\frac{dR}{du}\right) + \frac{1}{4}\Bigg\{\Bigg[\frac{[(u + a^2)(u + b^2) + abq]^2}{u\Delta_u} - \frac{a^2b^2}{u}\Bigg]\omega^2 - c_0\Bigg\}R = 0, \tag{10}$$

where $R \equiv R(u)$ and $\Delta_u \equiv \Delta_u(u)$. From now on, the dependence of the radial function will not be updated and the radial function will be denoted as R for each case.

We cast the equation in a form, which exhibits the singularity structure more transparently,

$$\frac{d}{du}\left(\Delta_u\frac{dR}{du}\right) + \frac{1}{4}\Bigg\{\Bigg[\frac{n_1}{\kappa_1^2(u - u_1)} + \frac{n_2}{\kappa_2^2(u - u_2)} + \frac{n_3}{\kappa_3^2(u - u_3)} + Gn_4\Bigg]\omega^2 - c_0\Bigg\}R = 0, \tag{11}$$

where the κ_i's are the surface gravities associated with each horizon located at u_i. The horizon coordinates u_i can be determined in terms of the bare black hole parameters via the horizon equation:

$$\Delta_u \equiv G(u - u_1)(u - u_2)(u - u_3), \tag{12}$$

which provides the following relations:

$$\sum_{i=1}^{3} u_i = -\left(\frac{1}{G} + a^2 + b^2\right), \tag{13}$$

$$\sum_{j<i=1}^{3} u_iu_j = \frac{1}{G}\left(a^2 + b^2 + a^2b^2G - 2m\right), \tag{14}$$

$$\prod_{i=1}^{3} u_i = -\frac{(q + ab)^2}{G}. \tag{15}$$

The surface gravities are given by [21],

$$\kappa_i = \frac{u_i^2\left[1 + G\left(2u_i + a^2 + b^2\right)\right] - (ab + q)^2}{\sqrt{u_i}\left[(u_i + a^2)(u_i + b^2) + abq\right]} \tag{16}$$

where $i = 1, 2, 3$ and they can be rewritten as,

$$\kappa_1 = \frac{G(u_1 - u_3)(u_1 - u_2)\sqrt{u_1}}{(u_1 + a^2)(u_1 + b^2) + abq}, \tag{17}$$

$$\kappa_2 = \frac{G(u_2 - u_3)(u_1 - u_2)\sqrt{u_2}}{(u_2 + a^2)(u_2 + b^2) + abq}, \tag{18}$$

$$\kappa_3 = \frac{G(u_2 - u_3)(u_1 - u_3)\sqrt{u_3}}{(u_3 + a^2)(u_3 + b^2) + abq}. \tag{19}$$

The n_i coefficients can be found after some algebra as,

$$n_1 = G(u_1 - u_2)(u_1 - u_3), \tag{20}$$

$$n_2 = -G(u_1 - u_2)(u_2 - u_3), \tag{21}$$

$$n_3 = G(u_1 - u_3)(u_2 - u_3), \tag{22}$$

$$n_4 = \frac{1}{G^2}. \tag{23}$$

It is important to note that the radial equation (11) has a pole structure, which is determined by the location of the three horizons and the residuum of the pole at each horizon is inversely proportional for the square of the surface gravity there. This result signifies the role that all three horizons have in determining the internal structure of such black holes. On the other hand, the resulting equation is Heun's equation and it does not have a compact form of a solution, not even in the case of the low-energy

ω of the scalar field. In the Appendix we further analyze the radial wave equation in terms of a dimensionless coordinate; we also perform a small G expansion, and an expansion in the near-extreme limit, suitable for the analysis in the subsequent sections.

A. The angular equation

Here we comment on the angular equation, which is for the general metric and taking $m_1 = m_2 = 0$ it is of the form,

$$\begin{aligned} & y^4 Y^2 \frac{d^2F}{dy^2} + y^3 Y \left(Y + y\frac{dY}{dy}\right)\frac{dF}{dy} \\ & - \left[(a^2b^2\omega^2 + c_0 y^2) y^2 Y \right. \\ & \left. + (a-y)^2(a+y)^2(b-y)^2(b+y)^2\omega^2\right] F = 0, \end{aligned} \tag{24}$$

where,

$$y = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}, \tag{25}$$

$$Y = -\frac{(1-Gy^2)(a^2-y^2)(b^2-y^2)}{y^2}. \tag{26}$$

Davis et al. have studied the angular equation without taking $m_1 = m_2 = 0$ and found that the solution can be given by general Heun's functions [24]. The structure of the equation does not change for $m_1 = m_2 = 0$ and the solutions can still be given in terms of general Heun's functions [30–33]. The angular equation in another parametrization was studied in [26] as a Sturm-Liouville problem and the eigenfunctions are the AdS modified spheroidal functions in five dimensions.

The angular part of the wave equation is independent of the non-extremality parameter, and thus the extreme and near-extreme cases, studied in the subsequent sections with general $m_{1,2}$, result in the same form of the angular equation as the general background. Therefore, for the sake of completeness we write below the angular equation with non-zero $m_{1,2}$ in the notation suitable for our subsequent analysis (in terms of u_1, G, a, b, q), as,

$$\begin{aligned} & Y\frac{d^2F}{dy^2} + \frac{1}{y}\left(y\frac{dY}{dy} + Y\right)\frac{dF}{dy} + \\ & \left(\frac{X}{\left[(b^2+u_1)(a^2+u_1)+abq\right]^2 y^4 Y}\right. \\ & \left. - 2G(u_1-u_3)c_{1,2}\right)F = 0, \end{aligned} \tag{27}$$

where,

$$\begin{aligned} X = & -\left\{m_2\, b\left(u_1+y^2\right)\left(-1+b^2G\right)a^4\right. \\ & + \left[Gm_1\left(u_1+y^2\right)b^4 + G\left(y^2m_2\,q + m_1u_1^2 - y^4m_1\right)b^2 - y^2\left(u_1\,Gm_1\,y^2 + Gm_1u_1^2 + m_2\,q\right)\right]a^3 \\ & + \left[G\left(u_1^2m_2 - y^4m_2 + y^2m_1\,q\right)b^2 + \left(-Gm_1\,q + m_2\right)y^4 - u_1^2m_2\right]ba^2 \\ & + \left[-m_1\left(u_1+y^2\right)b^4 + \left[\left(-Gm_2\,q + m_1\right)y^4 - m_1\,u_1^2\right]b^2 + y^2\left[\left(m_1\,u_1 + m_2\,q\right)y^2 + m_1\,u_1^2\right]\right]a \\ & \left. - by^2\left[\left(qm_1 + u_1\,m_2\,Gy^2 + Gm_2\,u_1^2\right)b^2 + \left(-u_1\,m_2 - qm_1\right)y^2 - u_1^2m_2\right]\right\}^2 \\ & - \left(u_1+y^2\right)u_1\,y^2\left\{\left[Gm_1\,ba^2 + m_2\left(-1+b^2G\right)a - m_1\,b\right]u_1 + m_2\left(-1+b^2G\right)a^3 + Gm_1\,b^3a^2 - m_1\,b^3\right\}^2 Y, \end{aligned} \tag{28}$$

and $c_{1,2}$ are separation constants, suitable for the parametrization in respective extreme and near-extreme cases. Note that c_0 separation constant in Eq.(24) is related to $c_{1,2}$ via a $2G(u_1 - u_3)$ factor.

The further analysis of the angular equation is beyond the scope of our study.

III. RADIAL EQUATION AND GREEN'S FUNCTION FOR THE EXTREME NEAR-HORIZON LIMIT

The extreme near-horizon scaling limit of the original metric was derived in Chow et al. [5]. The extreme limit implies,

$$u_2 = u_1, \tag{29}$$

and the scaling transformation of Chow et al. [5] on coordinates takes the form,

$$\begin{aligned} u &= u_1(1+\eta\rho)^2, \\ \hat{t} &= \beta t, \\ \hat{\phi}_1 &= \phi_1 + \Omega_1\hat{t}, \\ \hat{\phi}_2 &= \phi_2 + \Omega_2\hat{t}, \end{aligned} \tag{30}$$

where,

$$\beta = \frac{1}{4\pi u_1 \left(\frac{\partial T_H}{\partial u_1}\right)\eta}, \tag{31}$$

$$\Omega_1 = \frac{\Xi_a(au_1 + ab^2 + qb)}{(u_1 + a^2)(u_1 + b^2) + qab}, \tag{32}$$

$$\Omega_2 = \frac{\Xi_b(bu_1 + a^2b + qa)}{(u_1 + a^2)(u_1 + b^2) + qab}. \tag{33}$$

Here, Ω_1 and Ω_2 are the angular velocities and T_H is the Hawking temperature at the horizon u_1 given by Chow et al. [5].

The Vielbeine are obtained by taking the scaling limit $\eta \to 0$ and are given explicitly on [5]. Note that this specific scaling limit of the extreme solution "magnifies" the near-horizon region.

The solution Ansatz of the massless Klein-Gordon equation is of the following form,

$$\Phi = e^{-i\omega t + im_1\phi_1 + im_2\phi_2} R(\rho) F(y). \tag{34}$$

The angular part $F(y)$ satisfies Heun's equation (27). On the other hand, the radial equation for the extreme near-horizon scaling limit is given by,

$$\frac{d}{d\rho}\left(\rho^2 \frac{dR}{d\rho}\right) + \left\{\frac{[(m_1T_2 + m_2T_1)^2 - 2\,c_1\,\pi^2 T_1^2 T_2^2]\rho^2 + 2\,\pi\,T_1\,T_2\,(m_1T_2 + m_2T_1)\,\omega\rho + \pi^2 T_1^2 T_2^2 \omega^2}{4\rho^2\pi^2 T_1^2 T_2^2}\right\} R = 0, \tag{35}$$

where c_1 is the separation constant. T_1 and T_2 are the Frolov-Thorne temperatures given by Chow et al. [5],

$$T_1 = \frac{G\sqrt{u_1}(u_1 - u_3)\left[\left(a^2 + u_1\right)\left(b^2 + u_1\right) + abq\right]}{\pi\Xi_a\left[a\left(b^2 + u_1\right)^2 + bq\left(b^2 + 2u_1\right)\right]},$$

$$T_2 = \frac{G\sqrt{u_1}(u_1 - u_3)\left[\left(a^2 + u_1\right)\left(b^2 + u_1\right) + abq\right]}{\pi\Xi_b\left[b\left(a^2 + u_1\right)^2 + aq\left(a^2 + 2u_1\right)\right]}, \tag{36}$$

where we used a suitable notation for our purposes.

In the extreme limit, as $T_H = 0$ and G can be related to u_1, a, b, q as

$$G = \frac{(ab + q)^2 - u_1^2}{u_1^2(a^2 + b^2 + 2u_1)}. \tag{37}$$

G enters the equations via $\Xi_a = 1 - a^2 G$ and $\Xi_b = 1 - b^2 G$, and also in value of u_3 as,

$$u_3 = -\frac{(q + ab)^2}{G u_1^2}. \tag{38}$$

Note that the radial equation is written explicitly for non-zero values of $m_{1,2}$. Its solutions are special hypergeometric functions, i.e., Whittaker functions. Thus, the equation possesses the conformal $SL(2,\mathbf{R})^2$ symmetry. The explicit form of the radial solution can be written as a linear combination of two Whittaker functions M and W,

$$R = C_1'\, M\left(\frac{-\,iH_1}{2\,H_2}, \sqrt{\frac{1}{2}\left(c_1 + \frac{1}{2}\right) - \frac{H_1^2}{4H_2^2}}, \frac{i\omega}{\rho}\right) + C_2'\, W\left(\frac{-\,iH_1}{2\,H_2}, \sqrt{\frac{1}{2}\left(c_1 + \frac{1}{2}\right) - \frac{H_1^2}{4H_2^2}}, \frac{i\omega}{\rho}\right), \tag{39}$$

where C_1' and C_2' are constants and c_1 is the separation constant. Here we defined,

$$H_1 = m_1T_2 + m_2T_1, \tag{40}$$

$$H_2 = \pi\,T_1T_2. \tag{41}$$

The retarded Green's function can be obtained, analogous to four-dimensional Kerr backgrounds [16], as,

$$G_R \sim \frac{\mathcal{B}(m_1, m_2, \omega)}{\mathcal{A}(m_1, m_2, \omega)}, \tag{42}$$

where the coefficients $\mathcal{A}$ and $\mathcal{B}$ are determined by the asymptotic expansion of the radial solution, given in the form,

$$R^{in}_{m_1,m_2,\omega} \sim N\left(\mathcal{A}\rho^{-\frac{1}{2}+\beta} + \mathcal{B}\rho^{-\frac{1}{2}-\beta} + \text{higher order terms}\right). \tag{43}$$

where,

$$\beta = \sqrt{\frac{1}{2}\left(c_1 + \frac{1}{2}\right) - \frac{H_1^2}{4H_2^2}} - \frac{iH_1}{2H_2}. \tag{44}$$

The Green's function can be calculated accordingly (by expanding Whittaker functions),

$$G_R \sim \frac{(i\omega)^{\sqrt{2c_1-\frac{H_1^2}{H_2^2}+1}}\Gamma\left[-\sqrt{-\frac{H_1^2}{H_2^2}+2c_1+1}\right]\Gamma\left[\frac{1}{2}\left(\frac{iH_1}{H_2}+\sqrt{-\frac{H_1^2}{H_2^2}+2c_1+1}+1\right)\right]}{\Gamma\left[\sqrt{-\frac{H_1^2}{H_2^2}+2c_1+1}\right]\Gamma\left[\frac{1}{2}\left(\frac{iH_1}{H_2}-\sqrt{-\frac{H_1^2}{H_2^2}+2c_1+1}+1\right)\right]}. \tag{45}$$

It is interesting to observe that both Frolov-Thorne temperatures enter this Green's function, as they govern the two sectors of underlying the two-dimensional conformal field theory (CFT). These Green's functions can be further employed in the AdS/CFT context for the calculation of Lorentzian signature correlation functions for boundary field theory operators [34]. The retarded Green's functions for four-dimensional Kerr backgrounds were employed by Becker et al. [16] for such two-point and three-point correlation functions. The explicit form of retarded Green's function obtained in this section provides a starting point for quantitative studies of the boundary field theory correlators for the extreme charged AdS backgrounds.

IV. RADIAL EQUATION AND GREEN'S FUNCTION FOR THE NEAR-EXTREME NEAR-HORIZON LIMIT

In this section we derive the radial part of the wave equation in the near-extreme, near-horizon limit. For that purpose, we proceed with the expansion around the extreme limit $u_1 = u_2$, i.e. when the two horizons coincide. Thus, a deviation from extremality can be defined as,

$$u_2 = u_1(1 + p\eta), \tag{46}$$

where the small parameter $\eta \ll 1$. As we shall shortly see, p can be regarded as a (dimensionless) measure of the "effective near-extremality" and it does not have to be small. Furthermore, we introduce the scaling transformation of the metric coordinates (30). Now, the scaling limit $\eta \to 0$ signifies a "magnification" of near-extreme, near-horizon geometry.

In this limit the Vielbeine of the original general metric, which were given in [5], take the form,

$$e^0 = \sqrt{\frac{u_1+y^2}{2G(u_1-u_3)}}\sqrt{\rho(2\rho-p)}dt, \tag{47}$$

$$e^1 = \sqrt{\frac{u_1+y^2}{2G(u_1-u_3)}}\frac{d\rho}{\sqrt{\rho(2\rho-p)}}, \tag{48}$$

$$e^2 = \sqrt{\frac{Y}{u_1+y^2}}\left(\frac{a\tilde{e}_1\left(a^2+u_1\right)}{\left(a^2-b^2\right)\ \Xi_a}+\frac{b\tilde{e}_2\left(b^2+u_1\right)}{\left(b^2-a^2\right)\Xi_b}\right), \tag{49}$$

$$e^3 = \sqrt{\frac{u_1+y^2}{Y}}dy, \tag{50}$$

$$e^4 = \frac{ab}{\sqrt{u_1}y}\left(\frac{\tilde{e}_1\left(a^2-y^2\right)\left(b\left(a^2+u_1\right)\left(u_1+y^2\right)+aqy^2\right)}{ab\left(a^2-b^2\right)\left(u_1+y^2\right)\ \Xi_a} + \frac{\tilde{e}_2\left(b^2-y^2\right)\left(a\left(b^2+u_1\right)\left(u_1+y^2\right)+bqy^2\right)}{ab\left(b^2-a^2\right)\left(u_1+y^2\right)\Xi_b}\right), \tag{51}$$

where,

$$\begin{aligned} Y &= -\frac{(1-Gy^2)(a^2-y^2)(b^2-y^2)}{y^2}, \\ \tilde{e}_1 &= -\left(d\phi_1+\frac{\rho}{\pi T_1}dt\right), \\ \tilde{e}_2 &= -\left(d\phi_2+\frac{\rho}{\pi T_2}dt\right), \end{aligned} \tag{52}$$

and, T_i's are the Frolov-Thorne temperatures. This new explicit metric, obtained above by taking the near-extreme, near-horizon scaling limit, is now parameterized by the effective near-extremality parameter p. Of course, as $p \to 0$, one obtains the extreme near-horizon metric [5], which was employed in the previous section.

The Ansatz for the scalar function is chosen as,

$$\Phi = e^{-i\omega t+im_1\phi_1+im_2\phi_2}R(\rho)F(y). \tag{53}$$

The radial equation is given as,

$$\frac{d}{d\rho}\left[\rho\left(p-2\,\rho\right)\frac{dR}{d\rho}\right]+\frac{\left[c_2\pi^2T_1^2T_2^2(p-2\rho)\rho+\left(m_1T_2+m_2T_1\right)^2\rho^2+2\pi T_1T_2(m_1T_2+m_2T_1)\omega\rho+\pi^2T_1^2T_2^2\omega^2\right]}{\pi^2T_1^2T_2^2\left(p-2\,\rho\right)\rho}R=0, \tag{54}$$

and as $p \to 0$ this equation agrees exactly with the extreme case of the previous section. This equation is also a hypergeometric equation, and thus possesses the $SL(2,\mathbf{R})^2$ invariance associated with the conformal symmetry. The

solution can be written in terms of hypergeometric functions as,

$$\begin{aligned}
R = {} & (-p+2\,\rho)^{\frac{i[H_1 p+2\,\omega H_2]}{2H_2\,p}} \\
& \times\Bigg\{\rho^{\frac{-i\omega}{p}}\,C_1''\ {}_2F_1\left[\frac{H_2-\sqrt{(1+2c_2)\,H_2^2-H_1^2}+iH_1}{2H_2}\ ,\frac{H_2+\sqrt{(1+2c_2)\,H_2^2-H_1^2}+iH_1}{2H_2};\ \frac{p-2\,i\omega}{p};\ \frac{2\rho}{p}\right] \\
& +\rho^{\frac{i\omega}{p}}\,C_2''\ {}_2F_1\left[\frac{4\,i\omega H_2+\left(\sqrt{(1+2c_2)\,H_2^2-H_1^2}\right)p+[iH_1+H_2]\,p}{2H_2\,p}\right. \\
& \left.,\frac{4\,i\omega H_2-\left(\sqrt{(1+2c_2)\,H_2^2-H_1^2}\right)p+[iH_1+H_2]\,p}{2H_2\,p};\ \frac{p+2\,i\omega}{p};\ \frac{2\rho}{p}\right]\Bigg\}.
\end{aligned} \tag{55}$$

The retarded Green's function can furthermore be obtained by determining the ratio of the coefficients in the asymptotic expansion of the above solution as,

$$\begin{aligned}
G_R \sim {} & 2^{-\sqrt{2c_2-\frac{H_1^2}{H_2^2}+1}}\left(-\frac{1}{p}\right)^{-\sqrt{2c_2-\frac{H_1^2}{H_2^2}+1}} \\
& \times\frac{\Gamma\left[-\sqrt{-\frac{H_1^2}{H_2^2}+2c_2+1}\right]\Gamma\left[\frac{1}{2}\left(\frac{iH_1}{H_2}+\sqrt{-\frac{H_1^2}{H_2^2}+2c_2+1}+1\right)\right]\Gamma\left[\frac{1}{2}\left(-\frac{4i\omega}{p}+\sqrt{-\frac{H_1^2}{H_2^2}+2c_2+1}-\frac{iH_1}{H_2}+1\right)\right]}{\Gamma\left[\sqrt{-\frac{H_1^2}{H_2^2}+2c_2+1}\right]\Gamma\left[\frac{1}{2}\left(\frac{iH_1}{H_2}-\sqrt{-\frac{H_1^2}{H_2^2}+2c_2+1}+1\right)\right]\Gamma\left[\frac{1}{2}\left(-\frac{4i\omega}{p}-\sqrt{-\frac{H_1^2}{H_2^2}+2c_2+1}-\frac{iH_1}{H_2}+1\right)\right]}.
\end{aligned} \tag{56}$$

Note again that the retarded Green's function, while a bit more complex, carries an explicit dependence on both Frolov-Thorne temperatures as well as the effective non-extremality parameter p. Again this result is a starting point in the calculation of the Lorentzian signature correlation functions of the boundary conformal field theory operators [34] , analogous to calculations for the four-dimensional Kerr backgrounds in [16]. As $p \to 0$, the $\frac{4i\omega}{p}$ term dominates both in the numerator and in the denominator. Thus, these gamma functions cancel each other as the additional terms would become insignificant. Then, up to the factors in front of the gamma functions, we have the result we obtained in the extreme case.

V. CONCLUSIONS

The aim of this paper was to gain insight into the internal structure of rotating charged black holes in the asymptotically anti-de Sitter space-times. For this purpose we presented a comprehensive study of the massless scalar wave equation in the black hole background of the five dimensional minimal gauged supergravity. We focused on the radial part of such a separable wave equation, as the angular part was addressed in the past in [24, 26] in terms of general Heun's functions [30–33].

For the general black hole background the radial equation is Heun's equation, which can be cast, after a suitable coordinate transformation in a form whose structure is governed by the three poles associated with the locations of the three horizons and each residuum is inversely proportional to the square of the surface gravity there. The result signifies an importance of *all three horizons* when probing the internal structure of such black holes. It also complements a recent result of [27], where it was shown that the area product of the three horizons is quantized, an indication that all three horizons may play a role in microscopics of such general black holes.

We further focused on the study of the extreme and near-extreme limits of such black holes and the explicit solutions of the massless scalar wave equation in the near-horizon regime. For the extreme case such a scaling limit was studied in [5]. We find that the radial equation in this scaling limit is an equation for special hypergeometric (Whittaker) functions, and thus signifies the $SL(2,\mathrm{R})^2$ conformal invariance. The explicit retarded Green's function has a dependence on the two Frolov-Thorne temperatures, associated with the two sectors of the two-dimensional conformal theory.

In order to address the near-extreme near-horizon regime, we introduce a scaling limit, which corresponds to the same scaling transformations of the coordinates as in the extreme case; however, it also introduces an additional "effective near-extremality' parameter" which remains finite and dimensionless. The radial equation is an equation of general hypergeometric functions, and thus maintains the conformal invariance. The explicit retarded Green's function is now parametrized, in addi-

tion to the two Frolov-Thorne temperatures also by the effective non-extremality parameter.

These results provide a starting point for further quantitative studies of boundary conformal field theory. In particular, the explicit form of the retarded Green's function is a key ingredient in the calculation of the Lorentzian signature two-point and higher point correlation functions of the boundary field theory operators [34]. While progress along these directions has been made for the four-dimensional Kerr backgrounds [16], it would be important to pursue such calculations for the AdS black hole backgrounds, presented in this paper.

ACKNOWLEDGMENTS

TB would like to thank Mahmut Hortaçsu for various discussions and to the Department of Physics and Astronomy at the University of Pennsylvania for hospitality during the course of this work. MC would like to thank Gary Gibbons, Finn Larsen and Chris Pope for discussions. The research of TB is supported by TUBITAK, the Scientific and Technological Council of Turkey. MC is supported by the DoE Grant DOE-EY-76-02- 3071, the NSF RTG DMS Grant 0636606, the Fay R. and Eugene L. Langberg Endowed Chair, and the Slovenian Research Agency (ARRS).

Appendix: Radial equation, and small G and near-extreme limit expansion

For later purposes it turns out to be convenient to write the radial equation in terms of a dimensionless coordinate x and horizon locations x_i, which are related to u and u_i as:

$$u = P^{\frac{1}{3}} x, \tag{A.1}$$
$$u_i = P^{\frac{1}{3}} x_i, \tag{A.2}$$

where, $P \equiv (u_1-u_2)(u_1-u_3)(u_2-u_3)$. We note a useful property,

$$(x_1-x_2)(x_1-x_3)(x_2-x_3) = 1. \tag{A.3}$$

Furthermore, we apply another transformation:

$$x = \alpha y, \tag{A.4}$$
$$x_{1,2} = \alpha y_{1,2}, \tag{A.5}$$
$$x_3 = \frac{u_3}{P^{\frac{1}{3}}}, \tag{A.6}$$

where $\alpha \equiv \frac{u_1-u_2}{P^{\frac{1}{3}}}$. Then the radial equation becomes,

$$\begin{aligned}
&\partial_y\{(y-y_1)(y-y_2)[(u_1-u_2)y-u_3]\partial_y R\} \\
&+\frac{(u_1-u_3)(u_2-u_3)}{4}\Bigg\{\Bigg[\frac{1}{\kappa_1^2(y-y_1)((u_1-u_2)y_2-u_3)} \\
&\qquad -\frac{1}{\kappa_2^2(y-y_2)((u_1-u_2)y_1-u_3)} \\
&\qquad +\frac{1}{\kappa_3^2(y_1-y_2)((u_1-u_2)y-u_3)} \\
&\qquad +\frac{1}{G^2(u_1-u_3)(u_2-u_3)}\Bigg]\omega^2 \\
&-\frac{c_0}{G(u_1-u_3)(u_2-u_3)}\Bigg\}R = 0.
\end{aligned} \tag{A.7}$$

This equation is now in a suitable form for the expansion around small G as,

$$u_1 \approx u_{10} + u_{11}G + \mathcal{O}[G]^2, \tag{A.8}$$
$$u_2 \approx u_{20} + u_{21}G + \mathcal{O}[G]^2, \tag{A.9}$$
$$u_3 \approx -\frac{1}{G} - 2m + u_{31}G + \mathcal{O}[G]^2, \tag{A.10}$$
$$\kappa_{\substack{1\\2}}^2 \approx \kappa_{\substack{10\\20}}^2 + \kappa_{\substack{11\\21}}^2 G + \mathcal{O}[G]^2, \tag{A.11}$$
$$\kappa_3^2 \approx -G + \mathcal{O}[G]^2. \tag{A.12}$$

Now, we expand the radial function as,

$$R = R_0 + \varepsilon R_1 + \mathcal{O}[\varepsilon]^2. \tag{A.13}$$

Here, $\varepsilon \ll 1$ is a dimensionless quantity and can be chosen as,

$$\varepsilon = (u_1-u_2)G. \tag{A.14}$$

This may be regarded as a twofold approximation. One is $u_1 \approx u_2$ (near-extremality) and the other is $G \approx 0$ (small cosmological constant) case. The latter one should be valid for general values of other parameters.

The order of $\frac{1}{G^2}$ term is given by,

$$\frac{1}{4}\omega^2\left(1+\frac{1}{-y_1+y_2}\right)R_0 = 0, \tag{A.15}$$

and the y values can be chosen that,

$$y_{1,2} = \pm\frac{1}{2}, \tag{A.16}$$

which are the values given in [6].

The order of $\frac{1}{G}$ term is given by,

$$\begin{aligned}
&\partial_y\left[(y^2-\frac{1}{4})\partial_y R_0\right] + \frac{1}{4}\Bigg\{\Bigg[\frac{1}{\kappa_{10}^2(y-\frac{1}{2})} - \frac{1}{\kappa_{20}^2(y+\frac{1}{2})} \\
&-[2m+(u_{10}+u_{20})-(u_{10}-u_{20})y]\Bigg]\omega^2 \\
&\qquad -c_0\Bigg\}R_0 = 0.
\end{aligned} \tag{A.17}$$

Comparing this result with the radial equation given by Cvetič-Larsen [6], we see,

$$\begin{aligned}\Delta_{CL} &= (u_{10}-u_{20}) \\ &= \sqrt{[(a+b)^2-2(m-q)][(a-b)^2-2(m+q)]}, \\ M_{CL} &= -[2m+(u_{10}+u_{20})] \\ &= -[2m-(a^2+b^2-2m)] \\ &= a^2+b^2-4m, \end{aligned} \tag{A.18}$$

where the subscript "CL" denotes the related expression in the former paper. As $G\to 0$, the radial equation reduces to the one of [6], as expected.

In the expansion around the extreme limit (valid now for general values of G and remaining free parameters), we set,

$$u_2 = u_1(1+\varepsilon), \tag{A.19}$$

where $0\le\varepsilon\ll 1$ and $\varepsilon=0$ is the extreme limit.

The radial function is also expanded as,

$$R = R_0+\varepsilon R_1+\varepsilon^2 R_1+\mathcal{O}[\varepsilon^3]. \tag{A.20}$$

For the surface gravity associated with the outer horizon we have,

$$\begin{aligned}\kappa_1 &= \frac{Gu_1^2[u_1-u_1(1+\varepsilon)-u_3]+Gu_1^2u_3(1+\varepsilon)}{\sqrt{u_1}[(u_1+a^2)(u_1+b^2)+abq]} \\ &= \frac{\varepsilon Gu_1^2(u_3-u_1)}{\sqrt{u_1}[(u_1+a^2)(u_1+b^2)+abq]} \\ &\equiv \varepsilon\kappa_{11}, \end{aligned} \tag{A.21}$$

the surface gravity associated with the inner horizon is,

$$\begin{aligned}\kappa_2 &= \frac{Gu_1^2(1+\varepsilon)^2(\varepsilon u_1-u_3)+Gu_1^2(1+\varepsilon)u_3}{\sqrt{u_1(1+\varepsilon)}\{[u_1(1+\varepsilon)+a^2][u_1(1+\varepsilon)+b^2]+abq\}} \\ &\equiv -\varepsilon\kappa_{11}+\varepsilon^2\kappa_{22}+\mathcal{O}[\varepsilon^3], \end{aligned} \tag{A.22}$$

and for the surface gravity associated with the third horizon we have,

$$\begin{aligned}\kappa_3 &= \frac{Gu_3^2[u_3-u_1(2+\varepsilon)]+Gu_1^2u_3(1+\varepsilon)}{\sqrt{u_3}[(u_3+a^2)(u_3+b^2)+abq]} \\ &\equiv \kappa_{30}+\varepsilon\kappa_{21}. \end{aligned} \tag{A.23}$$

When we plug them into the radial equation, the ε^0 order gives an equation for R_0 only,

$$\frac{d}{du}\left[G(u-u_1)^2(u-u_3)\frac{dR_0}{du}\right]+\frac{1}{4}\left\{\left[\frac{Gu_1(\kappa_{11}u_1(u-u_3)+2\kappa_{22}(u-u_1)(u_1-u_3))}{\kappa_{11}^3(u-u_1)^2}+\frac{G(u_1-u_3)^2}{\kappa_{30}^2(u-u_3)}+\frac{1}{G}\right]\omega^2-c_0\right\}R_0=0, \tag{A.24}$$

and it is solved in terms of confluent Heun's functions (H_C),

$$\begin{aligned}R_0 = e^{-\frac{i\omega u_1}{2\kappa_{11}(u-u_1)}}\Big\{&C_1\,(u-u_3)^{\frac{i\omega}{2\kappa_{30}}}(u-u_1)^{-\frac{4\kappa_{30}+i\omega}{2\kappa_{30}}} \\ &\times H_C\left[\frac{-i\omega u_1}{\kappa_{11}\,(u_1-u_3)},\frac{i\omega}{\kappa_{30}},2,\frac{\omega^2u_1\,\kappa_{22}}{2\,(u_1-u_3)\,\kappa_{11}^3},\frac{[4\,(u_1-u_3)\,G^2-c_0G+\omega^2]\kappa_{11}^3-2\,u_1\,\omega^2G^2\kappa_{22}}{4\,(u_1-u_3)\,G^2\kappa_{11}^3},\frac{u-u_3}{u-u_1}\right] \\ &+C_2\,(u-u_3)^{-\frac{i\omega}{2\kappa_{30}}}(u-u_1)^{\frac{-4\kappa_{30}+i\omega}{2\kappa_{30}}} \\ &\times H_C\left[\frac{-i\omega u_1}{\kappa_{11}\,(u_1-u_3)},\frac{-i\omega}{\kappa_{30}},2,\frac{\omega^2u_1\,\kappa_{22}}{2\,(u_1-u_3)\,\kappa_{11}^3},\frac{[4(u_1-u_3)G^2-c_0G+\omega^2]\kappa_{11}^3-2\,u_1\,\omega^2G^2\kappa_{22}}{4\,(u_1-u_3)\,G^2\kappa_{11}^3},\frac{u-u_3}{u-u_1}\right]\Big\}, \end{aligned} \tag{A.25}$$

where C_1 and C_2 are constants.

In order to identify the near-horizon behavior of the radial equation in the near-extreme limit we will apply the following parametrization:

$$\omega \to \eta\omega_h, \tag{A.26}$$

$$u \to u_1(1+\eta\rho)^2, \tag{A.27}$$

and take the limit $\eta\to 0$. Then, for the leading order of η, Eq. (A.24) can be solved in terms of Bessel's functions,

$$\begin{aligned}R_0 = \frac{1}{\sqrt{\rho}}\Bigg[&C_1'\,J\left(-\frac{1}{2}\sqrt{1+\frac{c_0}{G\,(u_1-u_3)}},\frac{\omega_h}{4\kappa_{11}\,\rho}\right) \\ &+C_2'\,Y\left(-\frac{1}{2}\sqrt{1+\frac{c_0}{G\,(u_1-u_3)}},\frac{\omega_h}{4\kappa_{11}\,\rho}\right)\Bigg], \end{aligned} \tag{A.28}$$

where C_1' and C_2' are constants and c_0 is the separation constant. This solution formally agrees with the solution of the extreme case as the Whittaker functions are transformed into Bessel's functions as $m_1=m_2=0$. Since in the near horizon regime the equation is a hypergeometric equation, it possesses the $SL(2,\mathbf{R})^2$ conformal symmetry.

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